

# The choice of the symmetry group for the cuprates

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## Abstract

Following our recent conjecture to model the phenomena of antiferromagnetism and superconductivity by quantum symmetry groups, we discuss in the present note the choice of the classical symmetry group underlying the quantum group. Keeping in mind the degrees of freedom arising from spin, charge, and lattice we choose the classical group as  $SO(7)$ . This choice is also motivated to accommodate the several competing phases which are or may be present in these and related materials, such as stripe phase [mesoscopically ordered phase], Luttinger liquids, nearly antiferromagnetic Fermi liquids, charge-ordered Fermi liquids, glassy phase, stringy phase and perhaps more. The existence and the behavior of pseudo-gap and lattice distortion are also an important consideration. We have lumped the charge, spin and lattice-distortion ordering and other orderings into the pseudogap.

In a previous work one of us [1] have advanced the conjecture that one should attempt to model the phenomena of antiferromagnetism and superconductivity by using quantum symmetry group. Following this conjecture to model the phenomena of antiferromagnetism and superconductivity by quantum symmetry groups, three toy models were proposed [2], namely, one based on  $SO_q(3)$  the other two constructed with the  $SO_q(4)$  and  $SO_q(5)$  quantum groups. Possible motivations and rationale for these choices are were outlined. In [3] a model to describe quantum liquids in transition from 1d to 2d dimensional crossover using quantum groups was outlined.

In this short note we focus on the group choice since as mentioned before [1] that we feel that the quantum groups arising from the classical orthogonal groups, i.e.  $SO(N)$  are a good and worthwhile starting point, since they naturally incorporate the symmetry group of the insulating antiferromagnetic state and are naturally rich enough to accomodate quantum liquid behaviour.

The main purpose of this note is make a definite choice for the classical group underlying the quantum group. To this end we choose  $SO(7)$  for the purposes of this work. The choice of  $SO(8)$  is also tempting from the point of view of its octonians. We note that we may generally state that there are four major options confronting us towards the construction of a model of the cuprates, viz:

- The choice of the underlying Hamiltonian
- Symmetry Group
- Nature of symmetry:- for example classical or quantum
- Broken and unbroken symmeries.

We choose a Hubbard-BCS Hamiltonian, and  $SO_q(7)$  symmetry group. We take superconducting state to be dominated by a d-wave symmetry from experimental and theoretical

considerations. Contributions from a weak  $s$  wave component can be readily accommodated and limits may be placed on it from experiment. Current experiments indicate that the superconducting state is predominately  $d$  wave.

There are several reasons for choosing the classical group  $SO(7)$  which underlies directly the quantum group  $SO_q(7)$ :-

- As is well-known and already mentioned  $SO(3)$  [2] is a symmetry group for antiferromagnet insulator at the level of effective Hamiltonian.
- On the other hand effective Hamiltonian of a superconductor may be described by a  $U(1)$  nonlinear sigma model [XY model], for example it was indicated in Ref. [4] that the metal insulator transition may be described by the XY model.
- It was further pointed out in Ref. [5] that superconducting transition on the underdoped side of oxides be described by a renormalized classical model. In addition it is worth noting that  $SO(3)$  spin rotation and  $U(1)$  phase/charge rotation are symmetries of the microscopic t-J model.
- At the level of effective Hamiltonian one needs at least a  $SO(3)$  to be representative of charge ordering and local lattice distortions. This issue is a central one as recognized in [6] for example and not dealt with is charge inhomogeneity and phase separation.
- In theories based on magnetic interactions [7] for modelling of HTSC, it has been assumed that the  $CuO_2$  planes in HTSC materials are microscopically homogeneous. However, a number of experimental techniques have recently observed that the  $CuO_2$  are rather inhomogeneous, providing evidence for phase separation into a two component system. i.e. carrier-rich and carrier-poor regions [8]. In particular, extended x-ray absorption fine structure [EXAFS] demonstrated that these domains forms stripes of

undistorted and distorted local structures alternating with mesoscopic length scale comparable with coherence length in HTSC. The neutron pair distribution function of Egami et al. [9] also provides structural evidence for two component charge carriers. Other techniques also seem to point that below a certain temperature  $T^*$  the  $\text{CuO}_2$  planes may have ordered stripes of carrier-rich and carrier-poor domains [9]. The emergence of experimental evidence for inhomogeneous structure has led to renewal of interest, in theories of HTSC which are based on alternative mechanism, such as phonon scattering, the lattice effect on high  $T_c$  superconductivity [10–12,9]. Polarized EXAFS study of optimally doped  $\text{YBa}_2\text{Cu}_3\text{O}_y$  shows in-plane lattice anomaly [8] below a characteristic temperature  $T^{*\dagger}$  which lies above  $T_c$ , and close to the characteristic temperature of spin gap opening  $T^*$ . It is an interesting question if the in-plane lattice anomaly is related to the charge stripe or spin-phonon interaction. We note that it has been attempted in [13,14] to relate the spin gap observed in various experiments such as NMR, neutron scattering and transport properties to the short-range ordering of spin singlets.

Thus from the above we see that there are several phases, namely antiferromagnetism, superconductivity, charge ordering,... etc. If we assign  $SO(3)$  to antiferromagnetism,  $U(1)$  to superconductivity and  $SO(2)$  [or  $U(1)$ ] to the pseudogap we are naturally led to  $SO(7)$ . For one of the simplest group to embed  $SO(3) \times SO(2) \times U(1)$  or alternatively  $SO(3) \times U(1) \times U(1)$  is  $SO(7)$ . We note that this only one way of spontaneously breaking  $SO(7)$  there are several

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\*The following can be taken as a definition of  $T^*$ :  $T^*$  is an onset temperature of pseudogap opening in spin or charge excitation spectra.

$\dagger T^{*'}$  may be defined as follows:  $T^{*'}$  is an onset temperature of local phonon anomalies and  $T^{*' } < T^*$ .

others. The components of the seven-dimensional vector ( $\vec{\Phi} = (\phi_1, \dots, \phi_7)$ ) on which  $SO(7)$  acts can be assigned as follows:  $\phi_1 = \Delta_s + \Delta_s^\dagger$ , and  $\phi_2 = i(\Delta_s - \Delta_s^\dagger)$  for superconductivity,  $\phi_3$ ,  $\phi_4$ , and  $\phi_5$  for antiferromagnetism and  $\phi_6 = \Delta_p + \Delta_p^\dagger$ , and  $\phi_7 = i(\Delta_p - \Delta_p^\dagger)$  to represent the pseudogap. We have lumped the phenomena of charge ordering, spin ordering and lattice distortions into the pseudogap since we want to see if the ordering in these systems imply the existence of the pseudogap and temperature  $T^{*}$ . There are 21 symmetry generators for  $SO(7)$ . By breaking it down as above we have accounted for 5 generators, thus we are left with 16. These 16 generators can be interpreted on the basis of experiments on High  $T_c$  materials relating to spin [neutron scattering], charge ordering, and lattice-distortion [polarized EXAFS].

We now turn to classical  $SO(7)$  group. The generators of  $SO(7)$  can be written as<sup>†</sup>

$$[L_{ij}, L_{mp}] = i\delta_{im}L_{jp} - i\delta_{ip}L_{jm} + i\delta_{jp}L_{im} - i\delta_{jm}L_{ip} \quad (1)$$

We note that it is usual to denote the Lie algebra of  $SO(N)$  by  $so(N)$  which is the Lie algebra of all  $N \times N$  real antisymmetric matrices. It is useful to introduce  $N \times N$  real antisymmetric matrices  $L_{pm}$  defined by

$$(L_{pm})_{jk} = \delta_{pj}\delta_{mk} - \delta_{pk}\delta_{mj}, \quad p, m, j, k = 1, 2, \dots, N, \quad (2)$$

It immediately follows from Eq. 2 that for  $p \neq m$   $L_{pm}$  has zero elements everywhere except for an entry +1 in  $(p, m)$  position and -1 in  $(m, p)$  position,  $L_{pp} = 0$  and that  $L_{pm} = -L_{mp}$  [i.e. antisymmetric]. Due to the antisymmetry one can immediately see that the the generators  $L$  have which have  $7 \times 7 = 49$  elements are reduced to  $(49 - 7)/2 = 21$  elements. For  $SO(N)$

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<sup>†</sup>This relation is not particular to  $SO(7)$  as is true for all  $SO(N)$ . We are making this a point so that readers not familiar with group theory may not wrongly assume to the contrary.

the number of symmetry generators are  $N(N - 1)/2$  due to antisymmetry. The familiar rotation group in real three dimensions has 3 symmetry generators, this the reason why we need three Euler angles in 3 dimensions to parametrize rotations.

It can be shown from Eq. 2 that the generators satisfy the following commutation relation

$$[L_{ij}, L_{mp}] = i\delta_{im}L_{jp} - i\delta_{ip}L_{jm} + i\delta_{jp}L_{im} - i\delta_{jm}L_{ip} \quad (3)$$

which is nothing but Eq. 1 for the case  $N = 7$ .

As it is well-known any  $SO(N)$  represent orthogonal rotation in N-dimensions. Thus they leave the norm of any N-dimensional vector invariant. If we choose an order parameter  $\Phi$  with seven components viz  $\vec{\Phi} = (\phi_1, \dots, \phi_7)$  we know that quantity  $\phi_1^2 + \dots + \phi_7^2$  is invariant under  $SO(7)$ , we may normalize it and set it to unity. From standard group theory we know that or using the definition of  $L_{ij}$  we can immediately write down its commutation with any component of  $\vec{\Phi}$ , namely  $\phi_i$

$$[L_{pm}, \phi_i] = i\delta_{pi}\phi_m - i\delta_{mi}\phi_p \quad (4)$$

which is simply a statement of how rotation generator acts on the  $\vec{\Phi}$ .

The Kinetic energy of the system is also straightforward to write, since in analogy of spinning top from elementary physics it has the form of angular-momentum squared divided by the inertia  $J^2/2I$ . Thus we can write

$$\begin{aligned} \mathcal{L}_{K.E} &= \sum_{i,j} \frac{1}{2m_{ij}} L_{ij} L^{ij} \\ &= \sum_{i,j} \frac{1}{2m_{ij}} (L_{ij})^2 \end{aligned} \quad (5)$$

where  $m_{ij}$  represents the moment of inertia.

The potential energy term requires some discussion and there are several choice. We first note that it is straightforward to see that one can interpret  $\Phi$  as representing an order parameter and write immediately the Landau-Ginzburg expression for free energy

$$\mathcal{F} = m^2|\Phi|^2 + \lambda|\Phi|^4 \quad (6)$$

near the mean field transition as is familiar from ordinary field theory and statistical physics. In the phenomenological picture of phase transition the generators of  $SO(7)$  rotate the order parameter without changing its magnitude. Now symmetry will be spontaneously broken if a fixed direction is chosen, this happens when for example the system settles in a particular phase as is well-known. On the other hand we can explicitly break the  $SO(7)$  by introducing terms which don't respect the  $SO(7)$  symmetry, for example if we write the following term into the potential energy

$$\mathcal{L}_{P.E}^1 = -(a^2\phi_1^2 + b^2\phi_2^2). \quad (7)$$

This term clearly only includes unequal contributions from  $\phi_1^2$  and  $\phi_2^2$  and thus breaks  $SO(7)$  explicitly. It even breaks  $SO(2)$  symmetry for  $a^2 \neq b^2$ . If we chose  $a^2 = b^2$  this would respect  $SO(2)$  symmetry whilst breaking the remainder of  $SO(7)$  symmetry. We note that it is useful to know the subgroups for the purposes of symmetry breaking, hence we summarized some information for the groups of interest for us namely  $SO(7)$  and  $SO(8)$  in the appendix. Another type of term to first approximation that we must keep in the Lagrangian [see Appendix] is the velocity-dependent terms

$$\begin{aligned} \mathcal{L}_{P.E}^v &\sim \phi^i \phi_i \partial_a \phi_k \partial^a \phi^k \\ &\sim (v_{ik}^a)^2 \\ v_{ik}^a &= \phi_i \partial_a \phi_k - \phi_k \partial_a \phi_i. \end{aligned} \quad (8)$$

Collecting all the terms we can write the total Lagrangian as

$$\mathcal{L} = \sum_{i,j} \frac{1}{2m_{ij}} (L_{ij})^2 + \frac{\omega_{ij}}{2} (v_{ij}^a)^2 + \mathcal{L}_{ssb} + \mathcal{L}_{esb} \quad (9)$$

where ssb is shorthand for spontaneous symmetry breaking , esb for explicit symmetry breaking and  $\omega_{ij}$  are some ‘velocity’ parameters .

Even if we start with classical groups we end up with quantum groups if we examine their fixed points. This can be easily seen by examining the connection between Kac-Moody algebra and Quantum groups which leads to the important relation A11 [see Appendix]. This supports the conjecture in [1,2] that we should start with  $SO(N)_q$  since if we start with their classical counterparts and examine their fixed points we arrive at particular  $k$  values [ which are related to  $q$  values]. For example for  $SO(5)$  one arrives at  $SO(5)_{k=1}$  as is obvious from the discussion in the Appendix and as also noted in [16].

Another important point to note is that non-linear sigma models have connection with strings. In turn non-linear sigma models leads us naturally to the notion of noncommutativity via Kac-Moody algebra [quantum groups]. Yet another strong feature of quantum groups is that they unify classical Lie algebras and topology. In general sense it is expected that quantum groups will lead to a deeper understanding of the concept of symmetry in physics in particular condensed matter physics.

In conclusion, we propose  $SO(7)$  as the classical group underlying the quantum symmetry. Moreover  $SO(7)$  is interesting in its own right for phenomenological studies of HTSC materials. We have included the psuedogap into the  $SO(7)$  symmetry.

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## APPENDIX:

As is well-known that orthogonal rotation in  $N$ -dimension is specified by the  $SO(N)$  groups. By definition they leave the norm squared of  $N$  dimensional vector invariant. There is a distinction between when  $N$  is odd and even, i.e.  $SO(2n)$  and  $SO(2n+1)$ , in fact this is so in Cartan classification. We recall that in Cartan classification scheme groups are classified into the categories:  $A_n$ ,  $B_n$ ,  $C_n$ ,  $D_n$ ,  $G_2$ ,  $F_4$ ,  $E_6$ ,  $E_7$ , and  $E_8$ . For example  $SU(n+1)$  are in the category  $A_n$ ,  $SO(2n)$  are in  $D_n$ , and  $SO(2n+1)$  are in  $C_n$ .

The maximal subalgebra of classical simple Lie algebra of  $SO(7)$  reads:

$$\begin{aligned}
SO(7) &\supset SU(4), \\
SO(7) &\supset SU(2) \times SU(2) \times SU(2), \\
SO(7) &\supset Sp(4) \times U(1), \\
SO(7) &\supset G(2).
\end{aligned} \tag{A1}$$

We note that for  $Sp(4)$  the maximal subalgebra reads:

$$\begin{aligned}
Sp(4) &\supset SU(2) \times SU(2), \\
Sp(4) &\supset SU(2) \times U(1), \\
Sp(4) &\supset SU(2).
\end{aligned} \tag{A2}$$

$Sp(4)$  is isomorphic to  $SO(5)$ , and  $SO(4) \sim SU(2) \times SU(2)$ .  $SU(2) \supset U(1)$  and  $Sp(2)$ ,  $SO(3)$ , and  $SU(2)$  are all isomorphic.

The maximal subalgebra of classical simple Lie algebra of  $SO(8)$  is given by

$$\begin{aligned}
SO(8) &\supset SU(4) \times U(1), \\
SO(8) &\supset SU(2) \times SU(2) \times SU(2) \times SU(2), \\
SO(8) &\supset Sp(4) \times SU(2),
\end{aligned}$$

$$\begin{aligned}
SO(8) &\supset SU(3), \\
SO(8) &\supset SO(7).
\end{aligned}
\tag{A3}$$

In a simple sense one may say that non-linear sigma model is like a Taylor expansion in field theory. Let us explain what this means. We can write the action of string [1-dimensional] propagating in a manifold with metric  $G_{\mu\nu}$  as [15]

$$L \sim G_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu g^{ab} + \dots \tag{A4}$$

$g_{ab}$  is the two-dimensional metric generated by the ‘motion’ of string in the background manifold  $G_{\mu\nu}(X)$ . A crucial observation is that  $X$ ’s play a dual role of coordinate of the string in the background space and scalar field in the 2-dimensional space specified by the metric  $g_{ab}$  [i.e. the space generated by the motion of the string]. Different choices for the background metrics lead to different *conformal field theories*. Of interest to us is the choice that the string is propagating on a manifold specified by a Lie Group [for e.g.,  $SU(N)$ ,  $SO(N)$ , etc] in other words group manifold. We thus let  $g$  be an element of the Lie group. From Eq. A4 we can guess that a string propagating on this group manifold has an action of form

$$L \sim tr(\partial_a g^{-1} \partial^a g) \tag{A5}$$

where  $g$  is some function of the string field  $X$ . Simple differentiation gives

$$\partial_a g = \partial_a X_\mu f^{a\mu} \tag{A6}$$

for some function  $f$ , thus the metric  $G$  can be expressed in terms of  $f$ . The exact form of action is

$$\begin{aligned}
S &= \frac{1}{4\lambda^2} \int tr(\partial_a g^{-1} \partial^a g) + k\Gamma(g) \\
\Gamma(g) &= \frac{1}{24\pi} \int d^3 X \epsilon^{\alpha\beta\gamma} tr[(g^{-1} \partial^\alpha g)(g^{-1} \partial_\beta g)(g^{-1} \partial_\gamma g)]
\end{aligned}
\tag{A7}$$

where  $\Gamma(g)$  is the Wess-Zumino term which is integrated over 3-dimensional disk whose boundary is two-dimensional space. For  $k = 0$  it reduces to ordinary sigma-model, which is not conformally invariant [it is asymptotically free and massive]. For special values of  $k = 1, 2, 3, ..$  the theory becomes effectively massless and has an infrared-stable fixed point at the values of parameters  $\lambda$  and  $k$  related via

$$\lambda^2 = 4\pi/k \quad (\text{A8})$$

Thus at these special values of  $k$  we have a conformally invariant  $\sigma$  model where the theory is defined on the group manifold. This theory is called Wess-Zumino-Witten [WZW] model. The symmetry generators  $J$  satisfy a special case of Kac-Moody algebra, viz

$$[J_n^a, J_m^b] = f^{abc} J_{n+m}^c + \frac{1}{2} k n \delta^{ab} \delta_{n+m,0} \quad (\text{A9})$$

In Eq. A9 we note the following the generators  $J$  carry two indices, namely  $a, b, c, ..$  which are the Lie group indices and  $n, m, ..$  which arise in the decomposition of the generator  $J = J(z)$  in terms of its moments, viz,

$$J(z) = \sum_{n=-\infty}^{\infty} J_n z^{n-1}. \quad (\text{A10})$$

In some sense the Kac-Moody algebra smears the generators of ordinary Lie algebra around a circle or string.

Finally we recall [15,1] that

$$q \leftrightarrow e^{2\pi i/(k+2)} \quad (\text{A11})$$

If we make the above correspondence it can be shown by examining various identities of WZW model that the braiding properties of WZW model at level  $k$  are determined by the representation theory of quantum groups. As a trivial check if one sets  $q = 1$  in A11, which

is the limit in which quantum group reduces to the ordinary classical group, then the right-hand side of A11 we must set  $k \rightarrow \infty$ , which is precisely the limit in which Kac-Moody algebra reduces to ordinary classical algebra. We recall that the symmetry generators of the WZW model obey a special case of Kac-Moody algebra.

Non-linear sigma models have been extensively used in particle theory to describe interactions phenomenologically between strongly interacting particle. For example the Lagrangian for non-linear sigma-model for the special case of  $SU(2) \times SU(2)$  spontaneously broken to  $SU(2)$

$$\mathcal{L} = -\frac{1}{2} \frac{\partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi}}{(1 + \vec{\pi}^2/F^2)^2} \quad (\text{A12})$$

where the factor  $1/F$  acts as the coupling term that comes with the interaction of each additional pion. Expanding the expression in Eq. A12 and keeping only the first two terms we get

$$\mathcal{L} = -\frac{1}{2} \partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi} + (1/F^2) \vec{\pi}^2 \partial_\mu \vec{\pi} \partial^\mu \vec{\pi} + \dots \quad (\text{A13})$$

The first term is the simple kinetic energy term, the second being the potential energy term of the form  $\pi_i \pi^i \partial_\mu \vec{\pi}_j \partial^\mu \pi^j$  [in words it is a velocity dependent potential term]. We must also retain in general in our phenomenological modelling [as in this paper] of HTSC material keep such terms.

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